

# One-Parameter Squeezed Gaussian States of Time-Dependent Harmonic Oscillator and Selection Rule for Vacuum States

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## Abstract

By using the invariant method we find one-parameter squeezed Gaussian states for both time-independent and time-dependent oscillators. The squeezing parameter is expressed in terms of energy expectation value for time-independent case and represents the degree of mixing positive and negative frequency solutions for time-dependent case. A *minimum uncertainty proposal* is advanced to select uniquely vacuum states at each moment of time. We show that the Gaussian states with minimum uncertainty coincide with the true vacuum state for time-independent oscillator and the Bunch-Davies vacuum for a massive scalar field in a de Sitter spacetime.

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## I. INTRODUCTION

Harmonic oscillators have played many important roles in quantum physics, partly because they are exactly solvable quantum mechanically and partly because any system around an equilibrium can be approximated as a harmonic oscillator system. As a non-stationary system, a time-dependent quantum harmonic oscillator can also be exactly solved. One encounters typical time-dependent harmonic oscillators in a system of harmonic oscillators interacting with an environment or evolving in an expanding universe. In the former case, the harmonic oscillator system depends on time through parametric couplings to the environment. In the latter case, for instance, a massive scalar field, as a collection of harmonic oscillators when appropriately decomposed into modes, gains time-dependence from a time-dependent spacetime background. As a method to find the exact quantum states of a time-dependent harmonic oscillator, Lewis and Riesenfeld [1,2] have introduced an invariant, quadratic in momentum and position, which satisfies the quantum Liouville-Neumann equation. The exact quantum states are given by the eigenstates of this invariant up to some time-dependent phase factors. Since then there have been many variants and applications of the invariants and researches on the nature of the squeezed states of the vacuum states[3-29].

In this paper, first we circumvent technically the task of solving a time-dependent non-linear auxiliary equation in terms of which the quadratic invariant was expressed by Lewis and Riesenfeld [1,2], by finding a pair of first order invariants in terms of a complex solution to the classical equation and showing that the amplitude of the complex solution satisfies the auxiliary equation. By using the invariant method we find one-parameter squeezed Gaussian states which are symmetric about the origin. The squeezing parameter is determined by the energy expectation value for a time-independent oscillator and represents the degree of mixing positive and negative frequency solutions for a time-dependent oscillator. Second, we propose the *minimum uncertainty* as a rule to select uniquely the vacuum states for either time-independent and time-dependent oscillators. The Gaussian states with the minimum uncertainty have also the minimum energy expectation value at every moment

of time. The Gaussian states with minimum uncertainty coincide with the true vacuum state of time-independent oscillator and with the Bunch-Davies vacuum state for a minimal massive scalar field in a de Sitter spacetime.

The organization of this paper is as follows. In Sec. II we introduce a pair of first order invariants equivalent to the original quadratic invariant by Lewis and Riesenfeld and find one-parameter squeezed Gaussian states. In Sec. III we study the minimum uncertainty as a selection rule for vacuum states both for time-independent oscillator and for time-dependent oscillator.

## II. ONE-PARAMETER SQUEEZED GAUSSIAN STATES

First, we show the equivalence between the quadratic invariant introduced by Lewis and Riesenfeld [1,2] and a pair of first order invariants. Lewis and Riesenfeld let us solve the time-dependent Schrödinger equation in the Schrödinger-picture ( $\hbar = 1$ )

$$i\frac{\partial}{\partial t}\Psi(q, t) = \hat{H}(t)\Psi(q, t), \quad (1)$$

for a time-dependent harmonic oscillator of the form

$$\hat{H} = \frac{1}{2m_0}\hat{p}^2 + \frac{m_0\omega^2(t)}{2}\hat{q}^2. \quad (2)$$

Lewis and Riesenfeld introduced the invariant operator quadratic in position and momentum

$$\hat{I}(t) = \frac{1}{2m_0}\left[(\xi\hat{p} - \dot{\xi}\hat{q})^2 + \frac{1}{\xi^2}\hat{q}^2\right], \quad (3)$$

that satisfies the quantum Liouville-Neumann equation

$$i\frac{\partial}{\partial t}\hat{I} + [\hat{I}, \hat{H}] = 0. \quad (4)$$

Then  $\xi$  satisfies the auxiliary equation

$$\ddot{\xi} + \omega^2(t)\xi = \frac{1}{\xi^3}. \quad (5)$$

Instead of the quadratic invariant (3), let us consider a pair of the first order invariants [29]

$$\begin{aligned}\hat{A}(t) &= i(u^*(t)\hat{p} - m_0\dot{u}^*(t)\hat{q}), \\ \hat{A}^\dagger(t) &= -i(u(t)\hat{p} - m_0\dot{u}(t)\hat{q}).\end{aligned}\tag{6}$$

These operators satisfy the quantum Liouville-Neumann equation

$$\begin{aligned}i\frac{\partial}{\partial t}\hat{A}(t) + [\hat{A}(t), \hat{H}(t)] &= 0, \\ i\frac{\partial}{\partial t}\hat{A}^\dagger(t) + [\hat{A}^\dagger(t), \hat{H}(t)] &= 0,\end{aligned}\tag{7}$$

when  $u$  is a complex solution to the classical equation of motion

$$\ddot{u}(t) + \omega^2(t)u(t) = 0.\tag{8}$$

Imposing the commutation relation

$$[\hat{A}(t), \hat{A}^\dagger(t)] = 1,\tag{9}$$

as the annihilation and creation operators of a Fock space, is equivalent to requiring the Wronskian condition

$$m_0(\dot{u}^*(t)u(t) - u^*(t)\dot{u}(t)) = i.\tag{10}$$

To show the equivalence between the classical equation of motion (8) and the auxiliary equation (5), we write the complex solution in a polar form

$$u(t) = \frac{\xi(t)}{\sqrt{2m_0}}e^{-i\theta(t)}.\tag{11}$$

Then Eq. (10) becomes

$$\xi^2\dot{\theta} = 1,\tag{12}$$

and Eq. (8) equals to the auxiliary equation (5). Furthermore, one can rewrite the operators (6) as

$$\begin{aligned}\hat{A}(t) &= \frac{e^{-i\theta}}{\sqrt{2m_0}} \left[ \frac{m_0}{\xi} \hat{q} + i(\xi \hat{p} - m_0 \dot{\xi} \hat{q}) \right], \\ \hat{A}^\dagger(t) &= \frac{e^{i\theta}}{\sqrt{2m_0}} \left[ \frac{m_0}{\xi} \hat{q} - i(\xi \hat{p} - m_0 \dot{\xi} \hat{q}) \right]\end{aligned}\tag{13}$$

to show that

$$\hat{I}(t) = \hat{A}^\dagger(t) \hat{A}(t) + \frac{1}{2}.\tag{14}$$

The eigenstates of the invariant are the number states

$$|n, t\rangle = \frac{1}{\sqrt{n!}} \left( \hat{A}^\dagger(t) \right)^n |0, t\rangle,\tag{15}$$

where the vacuum state is defined by

$$\hat{A}(t)|0, t\rangle = 0.\tag{16}$$

Exact quantum states of the time-dependent harmonic oscillator are given explicitly by

$$|\Psi(t)\rangle = \sum_n c_n \exp\left(i \int \langle n, t | i \frac{\partial}{\partial t} - \hat{H}(t) | n, t \rangle\right) |n, t\rangle.\tag{17}$$

Second, we find one-parameter Gaussian states. For this purpose, we choose a specific positive frequency solution  $u_0$  to Eq. (8) such that

$$\text{Im}\left(\frac{\dot{u}_0(t)}{u_0(t)}\right) < 0,\tag{18}$$

and the Wronskian (10) is satisfied. We further require that  $u_0$  give the minimum uncertainty, which will be discussed in detail in the next section. Then any linear combination

$$u_\nu(t) = \mu u(t) + \nu^* u^*(t),\tag{19}$$

also satisfies the Wronskian condition (10) provided that

$$|\mu|^2 - |\nu|^2 = 1.\tag{20}$$

We now make use of the complex solution (19) to define

$$\begin{aligned}\hat{A}_\nu(t) &= i(u_\nu^*(t)\hat{p} - m_0\dot{u}_\nu^*(t)\hat{q}), \\ \hat{A}_\nu^\dagger(t) &= -i(u_\nu(t)\hat{p} - m_0\dot{u}_\nu(t)\hat{q}).\end{aligned}\tag{21}$$

Then the one-parameter Gaussian states can be found from the definition

$$\hat{A}_\nu(t)|0, t\rangle_\nu = 0,\tag{22}$$

whose coordinate representation are given by

$$\Psi_\nu(q, t) = \left(\frac{1}{2\pi u_\nu^*(t)u_\nu(t)}\right)^{1/4} \exp\left[i\frac{m_0\dot{u}_\nu^*(t)}{2u_\nu^*(t)}q^2\right].\tag{23}$$

Eq. (20) can be parameterized in terms squeezing parameters [31]

$$\mu \equiv \cosh r, \quad \nu \equiv e^{i\delta} \sinh r.\tag{24}$$

It follows readily that

$$\begin{aligned}\hat{A}_\nu(t) &= \tilde{\mu}\hat{A}(t) + \tilde{\nu}\hat{A}^\dagger(t), \\ \hat{A}_\nu^\dagger(t) &= \tilde{\nu}^*\hat{A}(t) + \tilde{\mu}^*\hat{A}^\dagger(t),\end{aligned}\tag{25}$$

where  $\tilde{\mu} = \mu$ , and  $\tilde{\nu} = e^{i(\delta+\pi)} \sinh r = -\nu$ . This can be rewritten as a unitary transformation

$$\hat{A}_\nu(t) = \hat{S}(z)\hat{A}(t)\hat{S}^\dagger(z),\tag{26}$$

where

$$\hat{S}(z) = \exp\left[\frac{1}{2}(z^*\hat{A}_0^2 - z\hat{A}_0^{\dagger 2})\right], \quad (z = re^{i(\delta+\pi)}),\tag{27}$$

is a squeeze operator [31]. Thus one sees that  $\Psi_\nu$  are the squeezed Gaussian states of  $\Psi_{\nu=0}$ .

It should be noted that

$$\xi^2(t) = 2m_0u_\nu^*(t)u_\nu(t),\tag{28}$$

indeed satisfies the auxiliary equation (5).

### III. SELECTION RULE FOR VACUUM STATES

In this section we study a selection rule for the vacuum states. It will be shown that the minimum uncertainty selects uniquely the vacuum states among the one-parameter Gaussian states in Sec. II. For the case of time-independent harmonic oscillator the minimum uncertainty state is the true vacuum state with the minimum energy expectation value. For the case of time-dependent harmonic oscillator the minimum state coincides with the Bunch-Davies vacuum state playing a particular role in quantum field theory in a curved spacetime. In this section we show that the one-parameter Gaussian states in Sec. II are parameterized by the energy expectation value and are the squeezed states of the true vacuum state. For time-dependent case we prove an inequality between the energy expectation value of a squeezed Gaussian state and that of the minimal squeezed Gaussian state.

#### A. Time-Independent Case: True Vacuum

For the case of a time-independent harmonic oscillator we can show that the squeezing parameter of one-parameter Gaussian states is nothing but the energy expectation value. The energy expectation value of the Hamiltonian with respect to the Gaussian state (23) is given by

$${}_{\nu}\langle 0, t | \hat{H} | 0, t \rangle_{\nu} = \frac{1}{4} \left( \dot{\xi}^2 + \omega^2 \xi^2 + \frac{1}{\xi^2} \right) \equiv \epsilon. \quad (29)$$

Equation (5) can be integrated to yield Eq. (29). We solve the integral equation (29) to obtain

$$\xi^2 = \frac{2\epsilon}{\omega^2} + \frac{2\epsilon}{\omega^2} \sqrt{1 - \frac{\omega^2}{4\epsilon^2}} \cos(2\omega t). \quad (30)$$

By solving (12) we get

$$\theta = \omega t. \quad (31)$$

We now compare the  $\xi^2$  of Eq. (30) with that obtained by solving directly Eq. (8). We choose the following specific solution to Eq. (8)

$$u_0(t) = \frac{1}{\sqrt{2m_0\omega}} e^{-i\omega t}, \quad (32)$$

and confine our attention to real  $\mu$  and  $\nu$ . It then follows that

$$\xi^2 = \frac{1}{\omega} (\mu^2 + \nu^2 + 2\mu\nu \cos(2\omega t)). \quad (33)$$

By comparing Eqs. (30) and (33) we find the squeezing parameter

$$\begin{aligned} \mu &= \sqrt{\frac{\epsilon}{\omega} + \frac{1}{2}}, \\ \nu &= \sqrt{\frac{\epsilon}{\omega} - \frac{1}{2}}. \end{aligned} \quad (34)$$

Thus we were able to express the squeezing parameters in terms of the energy expectation value.

We now look for the Gaussian state with the minimum uncertainty. The one-parameter Gaussian states have the uncertainty

$$(\Delta p)_\nu (\Delta q)_\nu = \frac{1}{2} (|\mu|^2 + |\nu|^2). \quad (35)$$

The minimal uncertainty state is obtained by  $\mu = 1$  and  $\nu = 1$  and the uncertainty is  $1/2$ .

This has also the minimum energy

$$\epsilon_{\min.} = \frac{\omega}{2}. \quad (36)$$

So the specific solution (32) corresponds to the minimum energy and the corresponding Gaussian state is the true vacuum state of the harmonic oscillator. Therefore, the Gaussian states we have found in Sec. II are the one-parameter squeezed states of the true vacuum state whose parameter is the energy expectation value.

## B. Time-Dependent Case: Bunch-Davies Vacuum

We now turn to the time-dependent case. In general, one can show the following inequality of the uncertainty relations with respect to  $\Psi_\nu$  and  $\Psi_0$



$$\begin{aligned}
(\Delta p)_\nu (\Delta q)_\nu &= m_0 \left( \dot{u}_\nu^*(t) \dot{u}_\nu(t) u_\nu^*(t) u_\nu(t) \right)^{1/2} \\
&\geq m_0 \left( \mu - |\nu| \right)^2 \left( \dot{u}_{\nu=0}^*(t) \dot{u}_{\nu=0}(t) u_{\nu=0}^*(t) u_{\nu=0}(t) \right)^{1/2} \\
&\geq m_0 \left( \dot{u}_{\nu=0}^*(t) \dot{u}_{\nu=0}(t) u_{\nu=0}^*(t) u_{\nu=0}(t) \right)^{1/2} \\
&= (\Delta p)_{\nu=0} (\Delta q)_{\nu=0}.
\end{aligned} \tag{37}$$

The equality of Eq. (37) holds when  $\mu = 1$  and  $\nu = 0$ . The energy expectation value similarly satisfies the inequality

$$\begin{aligned}
\langle \Psi_\nu | \hat{H} | \Psi_\nu \rangle &= m_0 \left( \dot{u}_\nu^*(t) \dot{u}_\nu(t) + \omega^2(t) u_\nu^*(t) u_\nu(t) \right) \\
&\geq \left( \mu - |\nu| \right)^2 m_0 \left( \dot{u}_{\nu=0}^*(t) \dot{u}_{\nu=0}(t) + \omega^2(t) u_{\nu=0}^*(t) u_{\nu=0}(t) \right) \\
&\geq m_0 \left( \dot{u}_{\nu=0}^*(t) \dot{u}_{\nu=0}(t) + \omega^2(t) u_{\nu=0}^*(t) u_{\nu=0}(t) \right) \\
&= \langle \Psi_{\nu=0} | \hat{H} | \Psi_{\nu=0} \rangle,
\end{aligned} \tag{38}$$

where the equality holds when  $\mu = 1$  and  $\nu = 0$ . What Eqs. (38) and (37) imply is that once we choose the Gaussian state with the minimum uncertainty and energy at each moment, all its squeezed Gaussian states have higher uncertainty and energy. However, it should be reminded that the energy expectation value for a time-dependent quantum system does not have an absolute physical meaning since it is not conserved. On the other hand, the quantum uncertainty still has some physical meanings even for the time-dependent quantum system in that it characterizes the very nature of quantum states. For this reason, we put forth the *minimum uncertainty* as the selection rule for the vacuum state for time-dependent system. From Eqs. (38) and (37), the vacuum state with the minimum uncertainty has also the minimum energy at every moment.

In order to show that the vacuum state with the minimum uncertainty coincides indeed with the well-known vacuum states we consider a minimal massive scalar field in the de Sitter spacetime. The de Sitter spacetime has the metric

$$ds^2 = -dt^2 + e^{2H_0 t} d\mathbf{x}^2, \tag{39}$$

where  $H_0$  is an expansion rate of the universe. When the massive scalar field is decomposed into Fourier modes, it has the Hamiltonian

$$H = \sum_{\mathbf{k},(\pm)} \frac{e^{-3H_0 t}}{2} (\pi_{\mathbf{k}}^{(\pm)})^2 + \frac{e^{3H_0 t}}{2} (m^2 + \mathbf{k}^2 e^{-2H_0 t}) (\phi_{\mathbf{k}}^{(\pm)})^2, \quad (40)$$

where  $u_{\mathbf{k}}^{(\pm)}$  denote the cosine- and sine-modes, respectively. Thus the massive scalar field system is equivalent to infinitely many harmonic oscillators both with a time-dependent mass and with time-dependent frequencies. Though the mass depends on time, all the previous results are valid only with the following modification

$$\begin{aligned} \hat{A}_{\mathbf{k}}^{(\pm)} &= i \left( u_{\mathbf{k}}^{(\pm)*}(t) \hat{\pi}_{\mathbf{k}}^{(\pm)} - e^{3H_0 t} \dot{u}_{\mathbf{k}}^{(\pm)*}(t) \hat{\phi}_{\mathbf{k}}^{(\pm)*} \right), \\ \hat{A}_{\mathbf{k}}^{(\pm)\dagger} &= -i \left( u_{\mathbf{k}}^{(\pm)}(t) \hat{\pi}_{\mathbf{k}}^{(\pm)} - e^{3H_0 t} \dot{u}_{\mathbf{k}}^{(\pm)}(t) \hat{\phi}_{\mathbf{k}}^{(\pm)} \right), \end{aligned} \quad (41)$$

where  $u_{\mathbf{k}}^{(\pm)}(t)$  satisfy the equations

$$\ddot{u}_{\mathbf{k}}^{(\pm)}(t) + 3H_0 \dot{u}_{\mathbf{k}}^{(\pm)}(t) + (m^2 + \mathbf{k}^2 e^{-2H_0 t}) u_{\mathbf{k}}^{(\pm)}(t) = 0. \quad (42)$$

It can be shown [30] that the specific solution in the Hankel function of the second kind

$$u_{\mathbf{k}}^{(\pm)}(t) = \left( \frac{\pi}{4H_0} \right)^{1/2} e^{-\frac{3}{2}H_0 t} H_{\chi}^{(2)}(z), \quad (43)$$

where

$$\chi = \left( \frac{9}{4} - \frac{m^2}{H_0^2} \right)^{1/2}, \quad z = \frac{k}{H_0} e^{-H_0 t}, \quad (44)$$

gives rise to the Gaussian state with the minimum uncertainty at each moment. Moreover, the Gaussian state has the uncertainty 1/2 at earlier times  $t \rightarrow -\infty$ . The vacuum state of the scalar field

$$|0, t\rangle_{\nu=0} = \prod_{\mathbf{k}, \pm} |0_{\mathbf{k}}^{(\pm)}\rangle_{\nu=0} \quad (45)$$

is indeed the Bunch-Davies vacuum state [32]. The general solution of the form (19) mixes the positive frequency solution  $u_{\mathbf{k}}^{(\pm)}$  with the negative frequency solution  $u_{\mathbf{k}}^{(\pm)*}$ .

#### IV. CONCLUSION

In this paper we have found the one-parameter squeezed Gaussian states for a time-dependent harmonic oscillator. It was found that the squeezing parameters can be expressed

in terms of energy expectation value and represents the degree of mixing of positive and negative frequency solutions. The *minimum uncertainty* is advanced as a selection rule for the vacuum state. We have illustrated the selection rule for the vacuum state by studying a time-independent harmonic oscillator and a minimal massive scalar field in a de Sitter spacetime. It was shown that the Gaussian states with the minimum uncertainty are the true vacuum state with the minimum energy for the time-independent harmonic oscillator and the Bunch-Davies vacuum state for the massive scalar field in the de Sitter spacetime.

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